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Pointwise dimensions on real Julia sets

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Abstract. From a formula for the pointwise dimensions of Bernoulli measures on real quadratic Julia sets, we deduce several aspects of their multifractal analysis. In particular, we prove a lower bound for the scaling function and give explicit formulae for its borders.

1. Introduction

Multifractal properties of ergodic measures on invariant sets of dynamical systems have already been studied in numerous works [2, 10, 7]. The purpose of this paper is to prove some complementary aspects for the special case of Bernoulli measures on real, totally disconnected quadratic Julia sets.

Starting from an expression for the pointwise dimensions in terms of the symbolic dynamics, we provide explicit formulae for the pointwise dimensions at periodic points of the dynamics. For one of the fixed points, this formula was already derived using other means, by Bessis *et al* [1]. Actually, the pointwise dimensions at the fixed points are of particular interest because they determine the borders of the scaling function of the maximal entropy measure. This follows from a lower bound on the scaling function expressed only in terms of the Lyapunov exponents of the family of Bernoulli measures on the Julia set. Its proof is given in section 5.

A physical application concerns the quantum diffusion of a localized wavepacket under the dynamics governed by a Julia matrix, notably the Jacobi matrix associated to the maximal entropy measure on the Julia set [4, 6]. The asymptotics of the growth exponents of this diffusion is given by the upper border of the scaling function [6]. More details are given in remark 4 below.

At this point let us note that our results, except for this application, can be directly transposed to other so-called *cookie cutter* dynamics generated by an application on the real line having two inverse branches [2]. For such dynamics, the self-similar invariant repellers can then be constructed just as the Julia sets (figure 1). Only the proof of proposition 1(i) has to be adapted to this situation.

2. Preliminaries

Let us first recall from [3] one construction of the real Julia set J, of the polynomial map $S(E) = E^2 - \lambda$, $E \in \mathbb{R}$ and $\lambda > 2$, as well as some of its properties. Starting from the real fixed points $E_\pm = (1 \pm \sqrt{1+4\lambda})/2$ of S, we construct the interval of zeroth generation as $I^0 = [-E_+, E_+]$. Given one-sided codes $\sigma \in \Sigma = \{-, +\}^{\times\mathbb{N}}$, the 2^n intervals of the nth

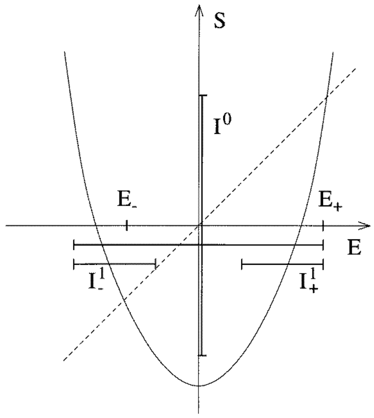

Figure 1. Schematic representation of the dynamics, the intervals of generation 0 and 1, as well as the fixed points E_+ and E_-.

generation are then given by $I^n_\sigma = S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}(I^0)$ where $S_\pm$ denote the two inverse branches of S. There exist constants $a < 1$ and $b > 0$ so that their lengths satisfy $|I^n_\sigma| \leqslant b\, a^n$. Then $J = \bigcap_{n\geqslant 0} \bigcup_{\sigma\in\Sigma} I^n_\sigma$ is a perfect, symmetric and S-invariant fractal set which is also the repeller of the map $S(z) = z^2 - \lambda$, $z \in \mathbb{C}$. The dynamical system (J, S) is conjugated to the shift on Σ by the coding map $E \in J \mapsto \sigma(E) \in \Sigma$. A given probability measure on Σ can be pulled back by the coding map in order to define a probability measure on J. In this paper, we shall mainly be concerned with the special class of shift-invariant, ergodic measures on Σ for which the σ_n, $n \in \mathbb{N}$, are independent random variables with the same distribution prob$\{\sigma_n = +\} = p$ and prob$\{\sigma_n = -\} = 1 - p$, $p \in [0, 1]$. Their pullback measures μ_p on J will be called Bernoulli measures with weight p. Note that μ_0 and μ_1 are the Dirac measures at the fixed points E_- and E_+, respectively. Let us point out that $\mu_{1/2}$ is the Frostman equilibrium measure as well as the maximal entropy measure on J.

3. Formula for the pointwise dimensions

Given any measure μ on $\mathbb{R}$, its lower- and upper-pointwise dimensions at $E \in \mathbb{R}$ are defined as

$$\underline{d}_\mu(E) = \liminf_{\epsilon\to 0} \frac{\log(\mu([E-\epsilon, E+\epsilon]))}{\log(\epsilon)} \qquad \overline{d}_\mu(E) = \limsup_{\epsilon\to 0} \frac{\log(\mu([E-\epsilon, E+\epsilon]))}{\log(\epsilon)}.$$

Whenever the limit $\epsilon \to 0$ exists, one also writes $d_\mu(E) = \underline{d}_\mu(E) = \overline{d}_\mu(E)$. For any measure μ supported on J, the pointwise dimensions are given by [2, 4, 7]

$$\underline{d}_\mu(E) = \liminf_{n\to\infty} \frac{-\frac{1}{n}\log(\mu(I^n_{\sigma(E)}))}{\frac{1}{n}\sum_{i=0}^{n-1}\log(|S'(S^{\circ i}(E))|)}$$
$$\overline{d}_\mu(E) = \limsup_{n\to\infty} \frac{-\frac{1}{n}\log(\mu(I^n_{\sigma(E)}))}{\frac{1}{n}\sum_{i=0}^{n-1}\log(|S'(S^{\circ i}(E))|)}.$$

It immediately follows from the Breiman–Shannon–McMillan theorem and Birkhoff's ergodic theorem that, whenever the measure μ is ergodic, the lower- and upper-pointwise dimensions coincide μ-almost surely and are μ-almost surely equal to the quotient of the dynamical entropy and the Lyapunov exponent of μ. Such measures with almost surely constant dimensions are called exactly scaling, and the almost sure value is sometimes called the information dimension. By the theory of Rogers and Taylor [8] and its dual [5], this almost-sure value is also equal to the Hausdorff and packing dimensions $\dim_{\mathrm{H}}(\mu)$ and $\dim_{\mathrm{P}}(\mu)$ of μ which are defined by the infimum of the Hausdorff (packing) dimensions of all Borel subsets $\Delta \subset \mathbb{R}$ satisfying $\mu(\Delta) = 1$. Furthermore [5], introducing the notations $\underline{d}^+_\mu(\Delta) = \mu\text{- esssup}_{E\in\Delta}\, \underline{d}_\mu(E)$ and $\overline{d}^+_\mu(\Delta) = \mu\text{- esssup}_{E\in\Delta}\, \overline{d}_\mu(E)$ for any Borel set $\Delta \subset \mathbb{R}$, the following identities hold whenever $\mu(\Delta) > 0$:

$$\begin{aligned} &\dim_{\mathrm{H}}(\{E \in \Delta | \underline{d}_\mu(E) = \underline{d}^+_\mu(\Delta)\}) = \underline{d}^+_\mu(\Delta) \\ &\dim_{\mathrm{P}}(\{E \in \Delta | \overline{d}_\mu(E) = \overline{d}^+_\mu(\Delta)\}) = \overline{d}^+_\mu(\Delta). \end{aligned} \tag{1}$$

For Bernoulli measures, the above formula for the lower-pointwise dimensions becomes

$$\underline{d}_{\mu_p}(E) = \liminf_{n\to\infty} \frac{-\rho_n(E)\log(p) - (1-\rho_n(E))\log(1-p)}{\frac{1}{n}\sum_{i=0}^{n-1}\log(|S'(S^{\circ i}(E))|)} \tag{2}$$

where $n\rho_n(E)$ is the number of + in the first n elements of the code $\sigma(E)$; a similar formula gives the upper pointwise dimensions. The other of the above statements can be resumed as follows:

$$\dim_{\mathrm{H}}(\mu_p) = \dim_{\mathrm{P}}(\mu_p) = \frac{\mathcal{E}(\mu_p)}{\Lambda(\mu_p)} = d_{\mu_p}(E) \qquad \mu_p\text{-a.s.} \tag{3}$$

where $\Lambda(\mu_p) = \int \mathrm{d}\mu_p(E)\log(2|E|)$ is the Lyapunov exponent of μ_p and $\mathcal{E}(\mu_p) = -p\log(p) - (1-p)\log(1-p)$ its dynamical entropy. We remark that $\dim(\mu_{1/2})$ can be shown to be a harmonic function of λ decreasing as $\log(4)/(\log(\lambda) + \mathrm{o}(1))$ [9].

As Hausdorff and packing dimensions coincide in all the cases we consider here, we suppress the corresponding index when calculating dimensions of sets or measures.

4. Local dimensions at periodic points

As the next application of formulae (2), we consider the pointwise dimensions at periodic points of S. The k-periodic points are solutions of the algebraic equation $S^{\circ k}(E) = E$. Each of them is associated to a code $(\eta_i)_{i=1\ldots k}$ of length k and we denote it by $E_{\eta_1\ldots\eta_k}$. It satisfies $S_{\eta_1}\ldots S_{\eta_k}(E_{\eta_1\ldots\eta_k}) = E_{\eta_1\ldots\eta_k}$. Then

$$d_{\mu_p}(E_{\eta_1\ldots\eta_k}) = \frac{-\rho_k(E_{\eta_1\ldots\eta_k})\log(p) - (1-\rho_k(E_{\eta_1\ldots\eta_k}))\log(1-p)}{\sum_{i=1}^k \log(|2S^{\circ i}(E_{\eta_1\ldots\eta_k})|)}.$$

These dimensions are constant on each orbit. Furthermore, the pointwise dimension of any preiterate of a periodic point is given by the dimension at the periodic point. This follows from the fact that the code of an mth preiterate can only differ from the code of the periodic point by the signs σ_i, $i = 1, \ldots, m$; this finite number of different values does not influence the value of the exponent in (2). Each set of preiterates is dense in J, however, as it is countable, its Hausdorff dimension is equal to zero.

In order to give more explicit formulae, let us first calculate the pointwise dimensions at the fixed points $E_\pm = (1 \pm \sqrt{1+4\lambda})/2$:

$$d_{\mu_p}(E_+) = \frac{-\log(p)}{\log(\sqrt{1+4\lambda}+1)} \qquad d_{\mu_p}(E_-) = \frac{-\log(1-p)}{\log(\sqrt{1+4\lambda}-1)}.$$

We remark that, for $p = \frac{1}{2}$, $d_{\mu_p}(E_+)$ has already been calculated in [1] by other means. It is worth noting that for p_λ determined as a solution of

$$p_\lambda + p_\lambda^{\beta_\lambda} = 1 \qquad \beta_\lambda = \frac{\log(\sqrt{1+4\lambda}-1)}{\log(\sqrt{1+4\lambda}+1)}$$

one has $d_{\mu_{p_\lambda}}(E_+) = d_{\mu_{p_\lambda}}(E_-)$. From $\beta_\lambda \in (\frac{1}{2}, 1)$ follows $p_\lambda \in ((3-\sqrt{5})/2, \frac{1}{2})$.

As a second explicit example, we consider the two-periodic points. Two are given by $E_\pm$, the other two by $E_{+-} = (-1+\sqrt{4\lambda-3})/2$ and $E_{-+} = (-1-\sqrt{4\lambda-3})/2$. Their pointwise dimension is

$$d_{\mu_p}(E_{-+}) = d_{\mu_p}(E_{+-}) = \frac{-\frac{1}{2}\log(p) - \frac{1}{2}\log(1-p)}{\log(2\lambda-2)}.$$

The following result shows that the pointwise dimensions at the fixed points are of particular interest because they give bounds for the dimensions at other points. Let us introduce the pointwise Lyapunov exponents at $E \in J$:

$$\underline{\Lambda}(E) = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(|S'(S^{\circ i}(E)|) \qquad \overline{\Lambda}(E) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(|S'(S^{\circ i}(E)|).$$

If the limit exists, we set $\Lambda(E) = \underline{\Lambda}(E) = \overline{\Lambda}(E)$.

Proposition 1. *Let* $\lambda_c = 2.32$.

(i) We have for any $E \in J$

$$\log(\sqrt{1+4\lambda}-1) \leqslant \overline{\Lambda}(E) \leqslant \log(\sqrt{1+4\lambda}+1)$$

where, for the first inequality, we suppose $\lambda > \lambda_c$ *to hold.*

(ii) For $p \geqslant \frac{1}{2}$, $\underline{d}_{\mu_p}(E) \geqslant d_{\mu_p}(E_+)$ *for all* $E \in J$.

(iii) For $\lambda > \lambda_c$, $\underline{d}_{\mu_{1/2}}(E) \leqslant d_{\mu_{1/2}}(E_-)$ *for all* $E \in J$.

The condition $\lambda > \lambda_c$ in (i) and (iii) can probably be removed, but it seems that our elementary proof does not allow one to treat λ in the interval $(2, \lambda_c)$, for which the non-hyperbolic character of the Julia sets is more difficult to trace. We further believe (ii) to be only partial. More precisely, we expect that $d_{\mu_p}(E_+)$ is the smallest pointwise dimension for all $p \geqslant p_\lambda$ and that $d_{\mu_p}(E_-)$ is the smallest dimension for $p \leqslant p_\lambda$ (compare remark 1 and figure 2 as an indication in this sense).

Proof of proposition 1. We note that $\overline{\Lambda}(E)$ is equal to $\log(2) + \limsup_{n\to\infty} \log(\prod_{i=0}^{n-1}|S^{\circ i}(E)|)/n$. Now $E_+ = \max_{E\in J}|E|$ and hence $\prod_{i=0}^{n-1}|S^{\circ i}(E_+)| = E_+^n$ is bigger or equal to $\prod_{i=0}^{n-1}|S^{\circ i}(E)|$ for any $E \in J$ and any $n \in \mathbb{N}$. Consequenly $\overline{\Lambda}(E) \leqslant \log(2)+\log(E_+)$, which is precisely the second inequality.

To prove the first one, it is sufficient to show that for all $E \in J$ there are infinitely many $n \in \mathbb{N}$ satisfying $\prod_{i=0}^{n-1}|S^{\circ i}(E)| \geqslant |E_-|^n$. For $n = 0$, this inequality is trivially satisfied. Suppose that it holds for $n-1$. If $|S^{\circ n}(E)| \geqslant |E_-|$, it also holds for n. Otherwise, some analysis of the function $g(E) = E \cdot |S(E)|$ on $\mathbb{R}_+$ shows $|S^{\circ n+1}(E) \cdot S^{\circ n}(E)| \geqslant \min\{E_+ \cdot S_+(-E_+), |E_-|^2\}$. Now $|E_-|^2 \leqslant E_+ \cdot S_+(-E_+)$ holds for all $\lambda > 2.62$ as numerical study of the algebraic expressions shows immediately. In this case, the above inequality holds also for $n+1$. Hence it is verified for at least every second n when $\lambda > 2.62$. In order to reach all $\lambda > 2.32$, one can refine the argument by treating a point in the left-most positive interval of the fourth-generation $[S_+(-E_+), S_+S_-S_+S_+(-E_+)]$ separately by considering the products $S^{\circ n}(E)S^{\circ n+1}(E)S^{\circ n+2}(E)$. Details are omitted.

For any E, the lower-pointwise dimensions of μ_p can be estimated from below:

$$\underline{d}_{\mu_p}(E) \geqslant \frac{-\log(1-p) + \log\left(\frac{1-p}{p}\right)\liminf_{n\to\infty}\rho_n(E)}{\overline{\Lambda}(E)}.$$

As $p \geqslant \frac{1}{2}$, $\log((1-p)/p) \leqslant 0$ and the factor $\liminf_{n\to\infty}\rho_n(E)$ in the denominator can be estimated by (1). The denominator is smaller or equal to $\log(|2E_+|)$ by (i). This gives

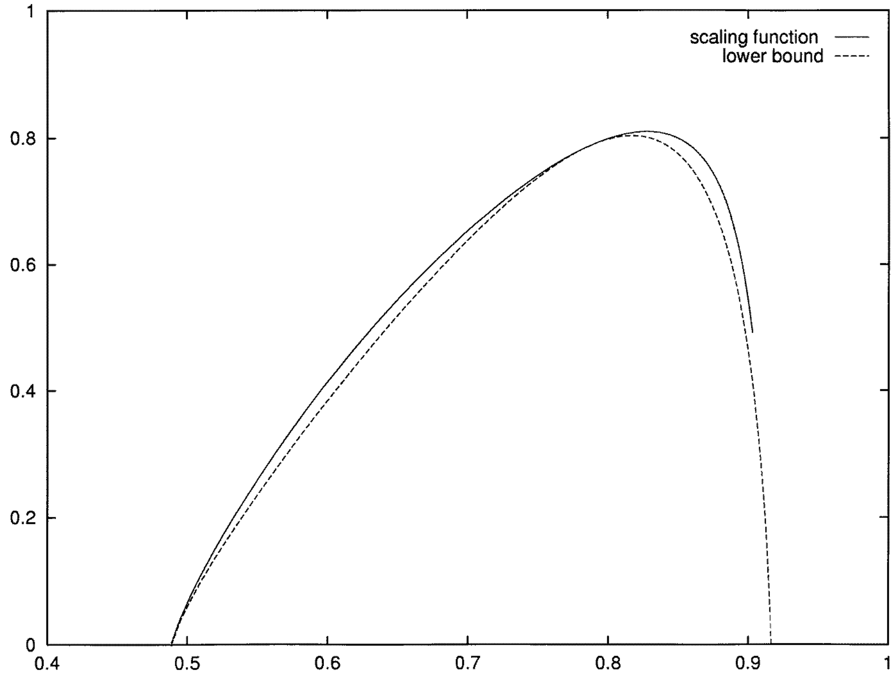

Figure 2. The scaling function calculated by use of the thermodynamic formalism and and its lower bound $\hat{f}$, given by (4) for $\lambda = 2.2$ and $p = \frac{1}{2}$.

(ii). Finally we note that the numerator in (2) is equal to log(2) if $p = \frac{1}{2}$. Therefore, $\underline{d}_{\mu_{1/2}}(E) = \log(2)/\overline{\Lambda}(E)$ and (i) implies (iii). □

5. Lower bound and borders of the scaling function

The scaling function of μ_p is defined to be $f_p(\alpha) = \dim(\{E \in J | d_{\mu_p}(E) = \alpha\})$. Equation (3) gives $f_p(\dim(\mu_p)) = \dim(\mu_p)$. We shall now bound f_p by a graph $\hat{f}_p$ given by the following one-parameter family of points in $\mathbb{R}^2$:

$$\left\{ \left(\frac{-r\log(p) - (1-r)\log(1-p)}{\Lambda(\mu_r)}, \dim(\mu_r) \right) \middle| r \in [0,1] \right\}. \tag{4}$$

Proposition 2. *$\hat{f}_p \leqslant f_p$, or more explicitly*

$$f_p\left(\frac{-r\log(p) - (1-r)\log(1-p)}{\Lambda(\mu_r)} \right) \geqslant \dim(\mu_r) \qquad \forall r \in [0,1].$$

This lower bound allows one to determine the borders of the scaling function.

Proposition 3. *(i) For $p \geqslant \frac{1}{2}$, $d_{\mu_p}(E_+) = \inf\{\alpha \in \mathbb{R} | f_p(\alpha) > 0\}$.*
(ii) For $\lambda > \lambda_c$, $d_{\mu_{1/2}}(E_-) = \sup\{\alpha \in \mathbb{R} | f_{1/2}(\alpha) > 0\}$.

Comments on these results follow their proofs.

Proof of proposition 2. For $r \in \{0, 1\}$, $\dim(\mu_r) = 0$ and there is nothing to prove. For $r \in (0, 1)$, let us denote by $\mathcal{F}(r)$, the set of points for which $(\rho_n(E), \sum_{i=1}^{n} \log(|2S^{\circ i-1}(E)|)/n)$

does not converge to $(r, \Lambda(\mu_r))$ in the limit $n \to \infty$. By the ergodic theorem, $\mu_r(\mathcal{F}(r)) = 0$. Next we introduce the sets

$$\mathcal{S}(r) = \left\{ E \in J\backslash\mathcal{F}(r) \,\middle|\, d_{\mu_p}(E) = \frac{-r\log(p) - (1-r)\log(1-p)}{\Lambda(\mu_r)} \right\}.$$

We shall prove that $\dim(\mathcal{S}(r)) = \dim(\mu_r)$ which already implies the desired result. Because the ergodic sums converge at the points in $\mathcal{S}(r)$, one has $\mathcal{S}(r) = \{E \in J\backslash\mathcal{F}(r) | d_{\mu_r}(E) = \dim(\mu_r)\}$. But now, since $J\backslash\mathcal{F}(r)$ has full μ_r-measure, the (Hausdorff or packing) dimension of $\mathcal{S}(r)$ is equal to $\dim(\mu_r)$ by equations (1) and (3). □

Proof of proposition 3. We first note that the application $p \in [0,1] \mapsto \mu_p$ is weakly continuous so that the family of Lyapunov exponents $\Lambda(\mu_p)$ is continuous in $p \in [0,1]$. According to proposition 1(ii), there are no pointwise dimensions smaller than $d_{\mu_p}(E_+)$. Hence $f_p(\alpha) = 0$ for $\alpha < d_{\mu_p}(E_+)$. On the other hand, the continuity of the Lyapunov exponents implies that, as $r \to 1$ in (4), the footpoint converges to $d_{\mu_p}(E_+)$. But for any $r < 1$, $\dim(\mu_r) > 0$ because $\mathcal{E}(\mu_r) > 0$ and $\Lambda(\mu_r) > 0$ in equation (3). Thus by proposition 2, $f(\alpha) > 0$ for $\alpha > \alpha_{\mu_p}(E_+)$. This proves (i). Item (ii) can be treated similarly by using propostion 1(ii). □

Remark 1. *The graphs $\hat{f}_p$ can easily be obtained numerically by calculating the Lyapunov exoponents $\Lambda(\mu_r)$ by the weak-limit representation [3] of μ_r by going up to $n \approx 10$. As r varies in the interval $[0,1]$, the corresponding points of $\hat{f}_p$ then give rise a continuous line because of the continuity of the Lyapunov exponents in r. Its end points are given by $(d_{\mu_p}(E_+), 0)$ and $(d_{\mu_p}(E_-), 0)$. However, the line may not be the graph of a function as can be seen in the example given in figure 2. In this figure, we have chosen p to be slightly smaller than p_λ for which the graph $\hat{f}_{p_\lambda}$ is a loop. Obviously, the graph in figure 2 cannot be a good lower bound for the scaling function. For comparison, we have also plotted the scaling functions calculated by use of the thermodynamic formalism [10]. This algorithm is proven ([7] and references therein) to give the exact curve of generalized dimensions $(D_p^q)_{q\in\mathbb{R}}$ and by Legendre transform one then obtains f_p. The scaling function f_p and its bound $\hat{f}_p$ always coincide at the information dimension, that is $f_p(\dim(\mu_p)) = \hat{f}_p(\dim(\mu_p)) = \dim(\mu_p)$ for all $p \in [0,1]$. Furthermore, the left border of f_p always coincides with the left-most point of $\hat{f}_p$ for all $p \in [0,1]$. Based on this observation, we speculated after proposition 1 that $d_{\mu_p}(E_+)$ is the smallest pointwise dimension for $p \geqslant p_\lambda$ and $d_{\mu_p}(E_-)$ for $p \leqslant p_\lambda$. This would imply corresponding results in proposition 3(i).*

Remark 2. *For $p = \frac{1}{2}$, $\hat{f}_{1/2}$ gives a tighter lower bound for $f_{1/2}$, but the curves do not coincide as the numerics in figure 3 show. Actually, we have the identity already remarked in [7]*

$$f_{1/2}\left(\frac{\log(2)}{\Lambda}\right) = \dim(\{E \in J \mid \Lambda(E) = \Lambda\}).$$

As a function of Λ, the right-hand side is called the Lyapunov spectrum [7]. Our bound reads

$$f_{1/2}\left(\frac{\log(2)}{\Lambda(\mu_r)}\right) \geqslant \dim(\{E \in J | \Lambda(E) = \Lambda(\mu_r) \qquad \textit{and} \qquad \lim_{n\to\infty} \rho_n(E) = r\}) = \dim(\mu_r).$$

From a measure theoretic point of view, the second condition is always verified when the first one is (notably, both ergodic limits exist on a set of full μ_r-measure). The situation is different from a dimensional point of view: imposing convergence of the second ergodic sum reduces the dimension except for $r = 0, \frac{1}{2}$ and 1.

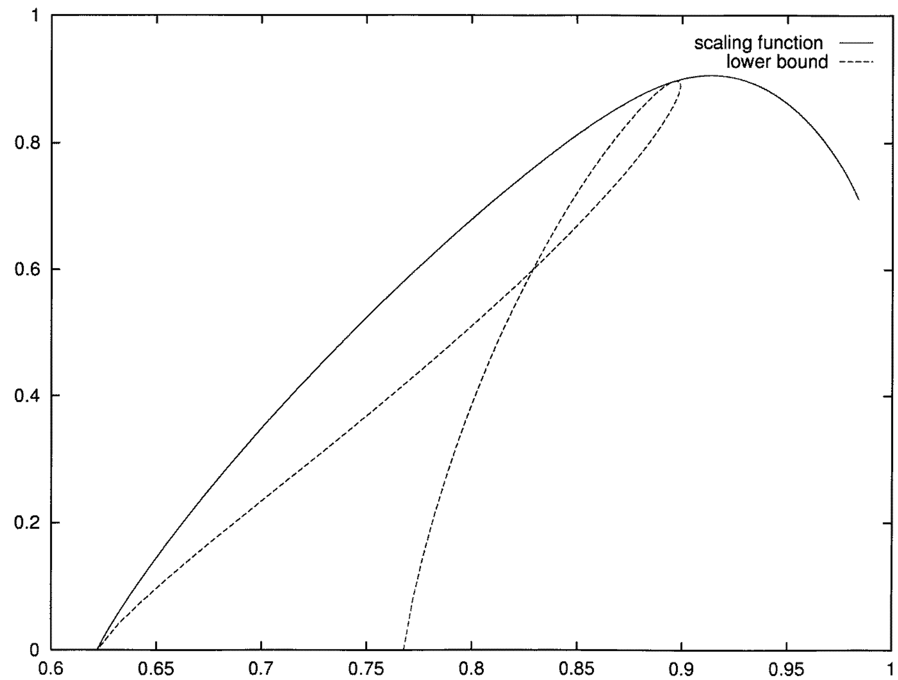

Figure 3. The scaling function calculated by use of the thermodynamic formalism and the graph of its lower bound (4) for $\lambda = 2.05$ and $p = 0.42$.

Remark 3. *The borders of* f_p *are equal to the asymptotics of the curve* D_p^q *in the limits* $q \to \pm\infty$ *which are numerically more delicate to determine by the thermodynamical formalism.*

Remark 4. *Let* $|n\rangle = P_n(E)$, $n \in \mathbb{N}$, *be the orthogonal polynomials associated to* $\mu_{1/2}$ *and* H *the corresponding Jacobi matrix. When studying quantum diffusion on the basis* $(|n\rangle)_{n\in\mathbb{N}}$ *of the dynamics governed by* H *[4–6], one is interested in the transport exponents* $\beta(q)$ *defined by*

$$\int_0^T \frac{\mathrm{d}t}{T} \sum_{n\geqslant 0} n^q |\langle n|\mathrm{e}^{-\mathrm{i}Ht}|0\rangle|^2 \underset{T\uparrow\infty}{\sim} T^{q\beta(q)}.$$

As argued in [6], one has $\beta(q) = D_{1-q}^{1/2}$ *for all* $q > 0$. *Thus proposition 3 implies*

$$\lim_{q\to\infty} \beta(q) = \frac{\log(2)}{\log(\sqrt{1+4\lambda} - 1)}.$$

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